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Integration

Estimation Formulas

For $f(x)$ on $[a,b]$, $\Delta x = \frac{b-a}{n}$:

Right-End Rectangles: $R_n = \Delta x \sum_{i=1}^{n} f[a + i\Delta x]$

Mid-Point Rectangles: $M_n = \Delta x \sum_{i=1}^{n} f[a + \left(i - \frac{1}{2}\right)\Delta x]$

Left-End Rectangles: $L_n = \Delta x \sum_{i=1}^{n} f[a + (i-1)\Delta x] = \Delta x \sum_{i=0}^{n} f[a + i\Delta x]$

Trapezoidal Rule: $T_n = \frac{1}{2}\Delta x[f(a) + 2\sum_{i=1}^{n-1} f[a + i\Delta x] + f(n)]$

$=\frac{1}{2}\Delta x(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-2} + 2y_{n-1} + y_n)$

Simpsons Rule: $S_n = \frac{1}{3}\Delta x[y_0 + 4y_1 + 2y_2 + \cdots + 4y_{n-2} + 2y_{n-2} + 4y_{n-1} + y_n$

Error, for $f(x)$ on $[a,b]$ with K_N being a number such that $\left|f^{(N)}(x)\right| \leq K_N$ for all $x \in [a,b]$ and n is the number of shapes used in the calculation:

Error(T_n) $\leq \frac{K_2(b-a)^3}{12n^2}$

Error(S_n) $\leq \frac{K_4(b-a)^5}{180n^4}$

Base Integration Rules

1. The Fundamental Theorem of Calculus, part I: $\int_a^b f(x)dx = F(b) - F(a)$
2. The Fundamental Theorem of Calculus, part II: $\frac{d}{dt}\int_a^t f(x)dx = f(t)$
3. Indefinite Integrals: $\int f(x)dx = F(x) + C$
4. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
5. $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx, \ a < b < c$
6. $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
7. $\int a\, f(x)dx = a\int f(x)dx$
8. $\int a\, dx = ax + C$
9. $\int x^n dx = \frac{1}{n+1}x^n + C, \ n \neq -1$
10. $\int x^{-1}dx = \ln|x| + C$
11. $\int e^x dx = e^x + C$
12. $\int a^x dx = \frac{1}{\ln(a)}a^x + C$
13. $\int \sin(x)\, dx = -\cos(x) + C$

14. $\int \cos(x)\,dx = \sin(x) + C$
15. $\int \sec^2(x)dx = \tan(x) + C$
16. $\int \csc^2(x)dx = -\cot(x) + C$
17. $\int \sec(x)\tan(x)dx = \sec(x) + C$
18. $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$
19. Substitution Method: for $\int u'(x)f(u(x))dx = \int f(u)du$
20. Integration by Parts: $\int u'v = uv - \int uv'$
21. $\int \tan(x)\,dx = \ln\ |\sec(x)| + C$
22. $\int \cot(x)\,dx = \ln\ |\sin(x)| + C$
23. $\int \sec(x)\,dx = \ln|sec(x) + tan(x)| + C$
24. $\int \csc(x)\,dx = \ln\ |\csc(x) - \cot(x)| + C$
25. $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$
26. $\int \frac{1}{|x|\sqrt{x^2-1}}dx = \text{arcsec}(x) + C$
27. $\int \frac{1}{x^2+1}dx = \arctan(x) + C$
28. $\int \ln(x)\,dx = xln|x| - x + C$

Hyperbolic Forms

29. $\int \sinh(x)\,dx = \cosh(x) + C$
30. $\int \cosh(x)\,dx = \sinh(x) + C$
31. $\int \tanh(x)\,dx = \ln[\cosh(x)] + C$
32. $\int \coth(x)\,dx = \ln|\sinh(x)| + C$
33. $\int \text{csch}(x)\,dx = \ln|\tanh\left(\frac{1}{2}x\right) + C$
34. $\int sech^2(x)\,dx = \tanh(x) + C$
35. $\int csc^2\text{h}(x)\,dx = -\coth(x) + C$
36. $\int \text{sech}(x)\,dx = \arctan[\sinh(x)] + C$
37. $\int \text{sech}(x)\tanh(x)\,dx = -\text{sech}(x) + C$
38. $\int \text{csch}(x)\coth(x)\,dx = -\text{csch}(x) + C$

Trigonometric Forms, Squares and Cubes

39. $\int \sin^2(x)\,dx = \frac{-1}{2}\sin(x)\cos(x) + \frac{1}{2}x + C = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
40. $\int \cos^2(x)\,dx = \frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}x + C = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
41. $\int \tan^2(x)\,dx = \tan(x) - x + C$
42. $\int \cot^2(x)\,dx = -\cot(x) - x + C$
43. $\int \sin^3(x)\,dx = \frac{-1}{3}\left(2 + \sin^2(x)\right)\cos(x) + C$

44. $\int \cos^3(x)\,dx = \frac{1}{3}\left(2+\cos^2(x)\right)\sin(x) + C$
45. $\int \tan^3(x)\,dx = \frac{1}{2}\tan^2(x) + \ln\,|\cos\,(x)| + C$
46. $\int \cot^3(x)\,dx = \frac{-1}{2}\cot^2(x) - \ln\,|\sin\,(x)| + C$
47. $\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln\,|\sec(x) + \tan\,(x)| + C$
48. $\int \csc^3(x)\,dx = \frac{-1}{2}\csc(x)\cot(x) + \frac{1}{2}\ln\,|\csc(x) - \cot\,(x)| + C$

Trigonometric Forms, General Forms

49. $\int \sin^n(x)dx = \frac{-1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx$
50. $\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin\,(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx$
51. $\int tan^n(x)dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)dx$
52. $\int \cot^n(x)dx = \frac{-1}{n-1}\cot^{n-1}(x) - \int \cot^{n-2}(x)dx$
53. $\int \sec^n(x)dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx$
54. $\int \csc^n(x)dx = \frac{-1}{n-1}\csc^{n-2}(x)\cot(x) + \frac{n-2}{n-1}\int \csc^{n-2}(x)dx$
55. $\int \sin(ax)\sin(bx)\,dx = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin\left((a+b)x\right)}{2(a+b)} + C$
56. $\int \cos(ax)\cos(bx)\,dx = \frac{-\sin((a-b)x)}{2(a-b)} - \frac{\cos\left((a+b)x\right)}{2(a+b)} + C$
57. $\int \sin^n(x)\cos^m(x)\,dx = -\frac{\sin^{n-1}(x)\cos^{m+1}(x)}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2}(x)\cos^m(x)dx$
$= \frac{\sin^{n+1}(x)\cos^{m-1}(x)}{n+m} + \frac{m-1}{n+m}\int \sin^n(x)\cos^{m-2}(x)dx$

Inverse Trigonometric Forms

58. $\int \arcsin(x)\,dx = xarcsin(x) + \sqrt{1-x^2} + C$
59. $\int \arccos(x)\,dx = xarccos(x) - \sqrt{1-x^2} + C$
60. $\int \arctan\,(x)\,dx = xarctan(x) - \frac{1}{2}\ln(1+x^2) + C$
61. $\int x^n\arcsin(x)\,dx = \frac{1}{n+1}\left[x^{n+1}\arcsin(x) - \int \frac{x^{n+1}}{\sqrt{1-x^2}}dx\right], n \neq -1$
62. $\int x^n\arccos(x)\,dx = \frac{1}{n+1}\left[x^{n+1}\arccos(x) + \int \frac{x^{n+1}}{\sqrt{1-x^2}}dx\right], n \neq -1$
63. $\int x^n\arctan(x)\,dx = \frac{1}{n+1}\left[x^{n+1}\arctan(x) - \int \frac{x^{n+1}}{1+x^2}dx\right], n \neq -1$

Forms involving $\sqrt{a^2 - x^2}, a > 0$

64. $\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{x^2}{2}\arcsin\left(\frac{x}{a}\right) + C$
65. $\int x^2\sqrt{a^2 - x^2} dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8}\arcsin\left(\frac{x}{a}\right) + C$
66. $\int \frac{\sqrt{a^2-x^2}}{x} dx = \sqrt{a^2 - x^2} - aln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$
67. $\int \frac{\sqrt{a^2-x^2}}{x^2} dx = \frac{-1}{x}\sqrt{a^2 - x^2} - \arcsin\left(\frac{x}{a}\right) + C$
68. $\int \frac{x^2}{\sqrt{a^2-x^2}} dx = \frac{-x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + C$
69. $\int \frac{1}{x\sqrt{a^2-x^2}} dx = \frac{-1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$
70. $\int \frac{1}{x^2\sqrt{a^2-x^2}} dx = \frac{-1}{a^2x}\sqrt{a^2 - x^2} + C$
71. $\int (a^2 - x^2)^{3/2} dx = \frac{-x}{8}(2x^2 - 5a^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8}\arcsin\left(\frac{a}{x}\right) + C$
72. $\int \frac{1}{(a^2-x^2)^{3/2}} dx = \frac{x}{a^2\sqrt{a^2-x^2}} + C$

Forms involving $\sqrt{x^2 - a^2}, a > 0$

73. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + C$
74. $\int x^2\sqrt{x^2 - a^2}\, dx = \frac{x}{8}(2x^2 - a^2)\sqrt{x^2 - a^2} - \frac{a^4}{8}\ln\left|x + \sqrt{x^2 - a^2}\right| + C$
75. $\int \frac{\sqrt{x^2-a^2}}{x} dx = \sqrt{x^2 - a^2} - a \cdot \arccos\left(\frac{a}{|x|}\right) + C$
76. $\int \frac{\sqrt{x^2-a^2}}{x^2} dx = \frac{-\sqrt{x^2-a^2}}{x} + \ln\left|x + \sqrt{x^2 - a^2}\right| + C$
77. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left|x + \sqrt{x^2 - a^2}\right| + C$
78. $\int \frac{x^2}{\sqrt{x^2-a^2}} dx = \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + C$
79. $\int \frac{1}{x^2\sqrt{x^2-a^2}} dx = \frac{\sqrt{x^2-a^2}}{a^2x} + C$
80. $\int \frac{1}{(x^2-a^2)^{3/2}} dx = \frac{-x}{a^2\sqrt{x^2-a^2}} + C$

Forms involving $\sqrt{a^2 + x^2}, a > 0$

81. $\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{x^2}{2}\ln\left|x + \sqrt{a^2 + x^2}\right| + C$
82. $\int x^2\sqrt{a^2 + x^2}\, dx = \frac{x}{8}(a^2 + 2x^2)\sqrt{a^2 + x^2} - \frac{a^4}{8}\ln\left|x + \sqrt{a^2 + x^2}\right| + C$
83. $\int \frac{\sqrt{a^2+x^2}}{x} dx = \sqrt{a^2 + x^2} - aln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C$
84. $\int \frac{\sqrt{a^2+x^2}}{x^2}\, dx = \frac{-\sqrt{a^2+x^2}}{x} + \ln\left|x + \sqrt{a^2 + x^2}\right| + C$

85. $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \ln\left|x+\sqrt{a^2+x^2}\right| + C$

86. $\int \frac{x^2}{\sqrt{a^2+x^2}}\,dx = \frac{x}{2}\sqrt{a^2+x^2} - \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| + C$

87. $\int \frac{1}{x\sqrt{a^2+x^2}}\,dx = \frac{-1}{a}\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C$

88. $\int \frac{1}{x^2\sqrt{a^2+x^2}}\,dx = \frac{-\sqrt{a^2+x^2}}{a^2x} + C$

89. $\int \frac{1}{(a^2+x^2)^{3/2}}\,dx = \frac{x}{a^2\sqrt{a^2+x^2}} + C$

Forms involving $a + bx$

90. $\int \frac{x}{a+bx}dx = \frac{1}{b^2}(a+bx-aln|a+bx|) + C$

91. $\int \frac{x^2}{a+bx}dx = \frac{1}{2b^3}[(a+bx)^2 - 4a(a+bx) + 2a^2\ln|a+bx|] + C$

92. $\int \frac{1}{x(a+bx)}dx = \frac{1}{a}\ln\left|\frac{x}{a+bx}\right| + C$

93. $\int \frac{1}{x^2(a+bx)}dx = \frac{-1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right| + C$

94. $\int \frac{x}{(a+bx)^2}dx = \frac{a}{b^2(a+bx)} + \frac{1}{b^2}\ln|a+bx| + C$

95. $\int \frac{1}{x(a+bx)^2}dx = \frac{1}{a(a+bx)} - \frac{1}{a^2}\ln\left|\frac{a+bx}{x}\right| + C$

96. $\int \frac{x^2}{(a+bx)^2}dx = \frac{1}{b^3}\left(a+bx-\frac{a^2}{a+bx}-2aln|a+bx|\right) + C$

97. $\int x\sqrt{a+bx}\,dx = \frac{2}{15b^2}(3bx-2a)(a+bx)^{\frac{3}{2}} + C$

98. $\int \frac{x}{\sqrt{a+bx}}dx = \frac{2}{3b^2}(bx-2a)\sqrt{a+bx} + C$

99. $\int \frac{x^2}{\sqrt{a+bx}}dx = \frac{2}{15b^2}(8a^2+3b^2x^2-4abx)\sqrt{a+bx} + C$

100. $\int \frac{1}{x\sqrt{a+bx}}dx = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right| + C,\ a > 0$

$= \frac{2}{\sqrt{-a}}\arctan\left(\sqrt{\frac{a+bx}{-a}}\right) + C, a < 0$

101. $\int \frac{\sqrt{a+bx}}{x}dx = 2\sqrt{a+bx} + a\int \frac{1}{x\sqrt{a+bx}}dx$

102. $\int \frac{\sqrt{a+bx}}{x^2}dx = \frac{-\sqrt{a+bx}}{x} + \frac{b}{2}\int \frac{1}{x\sqrt{a+bx}}dx$

103. $\int x^n\sqrt{a+bx}\,dx = \frac{2}{b(2n+3)}[x^n(a+bx)^{\frac{3}{2}} - na\int x^{n-1}\sqrt{a+bx}\,dx]$

104. $\int \frac{x^n}{\sqrt{a+bx}}dx = \frac{2x^n\sqrt{a+bx}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{x^{n-1}}{\sqrt{a+bx}}dx$

105. $\int \frac{1}{x^n\sqrt{a+bx}}dx = \frac{-\sqrt{a+bx}}{a(n-1)x^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{1}{x^{n-1}\sqrt{a+bx}}dx$

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Forms involving $\sqrt{2ax - x^2}, a > 0$

106. $\int \sqrt{2ax - x^2}\ dx = \frac{x-a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\arccos\left(\frac{a-x}{a}\right) + C$
107. $\int x\sqrt{2ax - x^2}\ dx = \frac{2x^2-ax-3a^2}{6}\sqrt{2ax - x^2} + \frac{a^3}{2}\arccos\left(\frac{a-x}{a}\right) + C$
108. $\int \frac{\sqrt{2ax-x^2}}{x}\ dx = \sqrt{2ax - x^2} + a \cdot \arccos\left(\frac{a-x}{a}\right) + C$
109. $\int \frac{\sqrt{2ax-x^2}}{x^2}\ dx = \frac{-2\sqrt{2ax-x^2}}{x} - \arccos\left(\frac{a-x}{a}\right) + C$
110. $\int \frac{1}{\sqrt{2ax-x^2}}\ dx = \arccos\left(\frac{a-x}{a}\right) + C$
111. $\int \frac{x}{\sqrt{2ax-x^2}}\ dx = -\sqrt{2ax - x^2} + a \cdot \arccos\left(\frac{a-x}{a}\right) + C$
112. $\int \frac{x^2}{\sqrt{2ax-x^2}}\ dx = \frac{-(x+3a)}{2}\sqrt{2ax - x^2} + \frac{3a^2}{2}\arccos\left(\frac{a-x}{a}\right) + C$
113. $\int \frac{1}{x\sqrt{2ax-x^2}}\ dx = \frac{-\sqrt{2ax-x^2}}{ax} + C$